THE CONDITIONS FOR THE EXISTENCE OF LIMIT CYCLES

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A qualitative investigation of a nonlinear differential equation of the second order is carried out. This equation, in particular, describes the motion of a particle along a closed curve subject to a pushing force. Under certain additional conditions the presence of a pushing force assures the existence of at least one stable limit cycle.

A simple illustrative example is a pendulum, subject to the action of a pushing force.

Consider the equation for the oscillations of a pendulum

$$\theta^{\cdot\cdot} + a\theta^{\cdot} + b\sin\theta = L + \delta M \tag{1}$$
$$\delta = 1 \text{ for } \theta^{\cdot} > 0, \qquad \delta = -1 \text{ for } \theta^{\cdot} \leqslant 0$$

where $\delta = 1$ for $\theta > 0$, $\delta = -1$ for $\theta < 0$ and a, b, L, M are positive constants such that

$$0 < L + M < b$$

holds.

The constant L in equation (1) corresponds to the presence of an external moment, the constant a characterizes the magnitude of the resistence of the medium and M is determined by the presence of a force pushing the pendulum in the direction of its motion.

Let θ_0 be the smallest positive angle satisfying the condition $\sin\theta_0 = (L + M)/b$. By the substitution $x = \theta - \theta_0$ equation (1) is reduced to the form

$$x^{\cdot\cdot} + ax^{\cdot} + f(x) = 0 \tag{2}$$

where

$$f(x) = b[\sin(x+\theta_0) - \sin\theta_0] \quad \text{for } x^{\cdot} > 0$$
$$f(x) = b[\sin(x+\theta_0) - \sin\theta_0] + 2M \quad \text{for } x^{\cdot} \leqslant 0$$

In the article under consideration oscillations determined by the equation

$$x^{\cdot\cdot} + R(x, x^{\cdot}) + f(x) = 0 \qquad (3)$$

are investigated, where

$$f(x) = f_1(x) \ \text{for} \ x^{\cdot} > 0, \qquad f(x) = f_2(x) \ \text{for} \ x^{\cdot} \leqslant 0$$

Conditions are imposed on the functions $R(x, x)$, $f_1(x)$ and $f_2(x)$, such that equation (3) becomes a generalization of equation (2).

Differential equation (3) is equivalent to the system of equations

$$x^{\cdot} = y, \qquad y^{\cdot} = -R(x, y) - f(x) \qquad (4)$$

The functions $f_1(x)$ and $f_2(x)$ are assumed to be continuous for all x, each continuously differentiable in the neighborhoods of its zeros. In order to avoid critical cases, it is assumed that the derivatives of the functions $f_1(x)$ and $f_2(x)$ do not vanish for the zeros of the functions themselves.

In addition, let the functions $f_1(x)$ and $f_2(x)$, for all x, satisfy the conditions

$$f_1(x + 2\pi) = f_1(x), \qquad f_2(x + 2\pi) = f_2(x)$$

and

$$f_1(0) = f_1(x_1) = f_2(x_2) = 0, \quad x_1 > 0, \quad x_2 < 0 \quad (x_1 - x_2 = 2\pi) \qquad (5)$$

$$f_2(\eta_0) = f_2(\eta_1) = f_2(\eta_2) = 0, \quad \eta_1 > 0, \quad \eta_2 < 0 \quad (\eta_1 - \eta_2 = 2\pi) \qquad (6)$$

Here x_1 and x_2 denote the roots of the function $f_1(x)$ nearest to $x = 0$, while η_1 and η_2 are the roots of the function $f_2(x)$ nearest to $x = \eta_0$, $xf_1(x) > 0$ in a neighborhood of $x = 0$, $(x - \eta_0)f_2(x) > 0$ in a neighborhood of $x = \eta_0$, and

$$J_1 = \int_0^{2\pi} f_1(x)\, dx \leqslant 0 \qquad (7)$$

In the present article we shall consider only one case of the mutual disposition of the roots of the functions $f_1(x)$ and $f_2(x)$, namely, we shall assume that the inequalities

$$x_2 < \eta_2 < \eta_0 < 0 < x_1 < \eta_1 \qquad (8)$$

hold.

Obviously, in the case of the example mentioned at the beginning of the article, the presence of the pushing force of the pendulum assures the above mentioned disposition of the roots of the functions $f_1(x)$ and $f_2(x)$. Sketches of the graphs of the functions $f_1(x)$ and $f_2(x)$ are given in Fig. 1. Notice that the graphs of these functions can, in general,

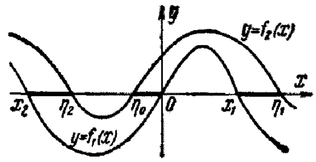

Fig. 1.

intersect. The integral

$$J_2(\eta_1, \eta_2) = \int_{\eta_2}^{\eta_1} f_2(x)\, dx \tag{9}$$

can assume any sign and may also be zero. For further investigations this fact will be of essential importance.

We shall assume that the function $R(x, y)$ is continuous on the whole xy-plane, continuously differentiable in the neighborhoods of the points $(\eta_0, 0)$, $(0, 0)$, $(x_1, 0)$, $(\eta_1, 0)$ and such that the conditions

$$R(x + 2\pi, y) = R(x, y)$$

$$R(x, y) \text{ increases with } y \tag{10}$$

$$\lim R(x, y) > 0 \quad \text{as } y \to \infty, \qquad R(x, 0) \equiv 0, \qquad \lim_{y\to-\infty} R(x, y) < 0 \quad \text{as } y \to -\infty$$

are satisfied.

Equation (3), satisfying the conditions (5) to (8), in general, describes the motion of the particle along a certain closed curve, subject to the action of a pushing force.

The aim of the present paper is a qualitative investigation of system (4) under the assumptions (5) to (8), i.e. the consideration of the possible different dispositions of the integral curves of this system and the study of its limit cycles.

From the assumptions concerning the function $f(x)$ it follows that the trajectories of the system (4) for the half-plane $y > 0$ are determined by the system of differential equations

$$x^{\cdot} = y, \qquad y^{\cdot} = -R(x, y) - f_1(x) \tag{11}$$

while for the half-plane $y < 0$ by the system

$$x^{\cdot} = y, \qquad y^{\cdot} = -R(x, y) - f_2(x) \tag{12}$$

The possible disposition of the trajectories of the system (11) under the assumptions (5), (10) has been investigated fully in an earlier paper [1]. If the condition $J_2(\eta_2, \eta_1) \leqslant 0$ is satisfied for system (12) also the disposition of its trajectories can easily be established by means of Theorem 1 of the same paper [1]. In order to apply the results of the above mentioned paper to system (12) for the case $J_2(\eta_2, \eta_1) > 0$, it is sufficient to carry out a substitution of variables, by replacing y by $-y$ and x by $\eta_0 - x$. Introduce the notations

$$f_2(\eta_0 - x) = F(x), \qquad -R(\eta_0 - x, -y) = R_1(x, y)$$

Then it is easy to see that equation

$$\frac{dy}{dx} = \frac{-R_1(x, y) + F(x)}{y}$$

which is equivalent to the system of equations (12), satisfies the assumptions of Theorem 1 of article [1], i.e. the results established in that theorem can be applied anew.

In order to investigate the disposition of the trajectories of system (4) on the whole plane, it is sufficient to "paste together" along the x-axis the trajectories of system (11) for the half-plane $y > 0$ and the corresponding trajectories of system (12) for the half-plane $y \leqslant 0$. Let us elaborate on this in detail.

The phase trajectories of system (4) are determined by the differential equation

$$\frac{dy}{dx} = -\frac{R(x, y) + f(x)}{y} \tag{13}$$

where the functions $R(x, y)$ and $f(x)$ satisfy conditions (5) to (7).

We shall consider the phase space of system (4) as being developed on the xy-plane [1]. Because of the periodicity with respect to x of $f_1(x)$, $f_2(x)$ and $R(x, y)$, the disposition of the integral curves in the xy-plane will be the same in all strips of width 2π and parallel to the y-axis.

Therefore, it is sufficient to investigate the disposition of the integral curves of equation (12), for example, in the strip

$$x_2 \leqslant x \leqslant x_1 \quad \text{for } y > 0, \qquad \eta_2 \leqslant x \leqslant \eta_1 \quad \text{for } y \leqslant 0$$

Of essential significance for the disposition of the integral curves of system (4) are the singular points of systems (11) and (12), the coordinates of which are found from the equations $f_1(x) = 0$, $f_2(x) = 0$, $y = 0$. In order that the motions of system (4) have an oscillating character, assume that the following conditions are satisfied:

$$[R_y'(\eta_0, 0)]^2 < 4f_2'(\eta_0), \qquad [R_y'(0, 0)]^2 < 4f_1'(0).$$

The singular points of systems (11) and (12) decompose the interval $[x_2, \eta_1]$ into several segments, of which the three segments $[x_2, \eta_2]$,

$[\eta_0, 0]$ and $[x_1, \eta_1]$ consist in their entirety of unstable points (Fig. 2). These segments will be called unstable segments of system (4).

We shall study the disposition of the separatrices of system (4), and also of the several needed integral curves of this system by means of the isoclinics of equation (13). Isoclinics of inclination zero are the curves determined by the equations

$$f_1(x) + R(x, y) = 0 \quad \text{for } y > 0, \qquad f_2(x) + R(x, y) = 0 \quad \text{for } y \leqslant 0$$

Due to the assumptions mentioned below regarding the functions $f_1(x)$, $f_2(x)$ and $R(x, y)$, these equations uniquely determine the dependence of the ordinates of the points of the isoclinic as functions of their abscissas

$$y = \Phi_1(x) \text{ for } y > 0, \qquad y = \Phi_2(x) \text{ for } y \leqslant 0. \tag{14}$$

Moreover, as it is easy to establish, the zeros and the intervals in which the functions $\Phi_1(x)$ and $\Phi_2(x)$ have a constant sign coincide with the zeros and intervals in which the functions $f_1(x)$ and $f_2(x)$, respectively, have a constant sign. The isoclinics of zero inclination (14),

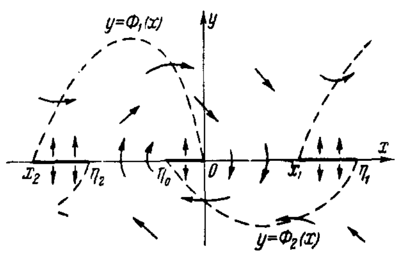

Fig. 2.

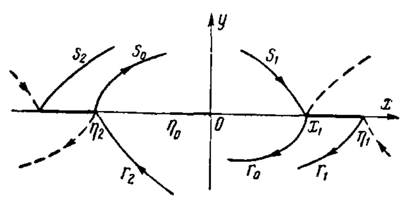

Fig. 3.

together with the isoclinic of vertical inclination $y = 0$ of the integral curves, decompose the xy-plane into regions in which the derivative of the integral curves of system (4) has constant sign. Therefore, the direction field of the system can be easily constructed (Fig. 2).

The study of the direction field of system (4), satisfying assumptions (5) to (10), and of the nature of the singular points of systems (11) and (12), shows us that the presence of unstable segments of system (4), in comparison with the previously considered case [1], creates a larger number of possible qualitative pictures for the disposition of the integral curves of system (4). In particular, the presence of the unstable segment $[\eta_0, 0]$ may imply the existence of a stable limit cycle, embracing this segment and corresponding to the periodic solution $X(t)$ of equation (3).

If the conditions (5) and (8) are also satisfied, it is possible that system (4) possesses periodic solutions $Y_1(x)$ and $Y_2(x)$, corresponding to the limit cycles embracing the cylinder of the phase surface of the system and satisfying for all x the conditions

$$\begin{gathered} Y_i(x+2\pi) = Y_i(x) \qquad (i=1,2) \text{ for all } x \\ \int_0^{2\pi} [f_i(x) + R(x, Y_i(x))]\, dx = 0, \qquad Y_1(x) \geqslant 0, \qquad Y_2(x) \leqslant 0 \end{gathered} \tag{15}$$

The functions $Y_1(x)$ and $Y_2(x)$ can vanish only at the points which correspond to the abscissa of the unstable singular points of systems (11) and (12), respectively.

Let us denote the separatrices of system (4), and also some of their continuations, by means of integral curves as in Fig. 3. For the convenience of the exposition the latter will also be called separatrices of system (4).

The varieties of the qualitative picture for the disposition of the integral curves of the system under consideration are determined by the various combinations of the existence and nonexistence of the periodic solutions $X(t)$, $Y_1(x)$, $Y_2(x)$ of equations (3) and (13), respectively, as well as by the various possible mutual dispositions of the separatrices s_1, s_2, r_1, r_2, r_0, s_0.

Analogous to Theorem 1 of article [1], a theorem can be formulated and proved to the effect that system (4) under the assumptions (5) to (7) possesses five and only five possible varieties for the qualitative picture of the disposition of the integral curves.

The necessary and sufficient conditions for the existence and nonexistence of periodic solutions with respect to x are given in Theorem 1 of the previously mentioned paper [1]. So the fulfillment of the inequality

$$s_1(0) > s_2(0) \tag{16}$$

is a necessary and sufficient condition for the nonexistence of the solution $Y_1(x)$. On the other hand, for the existence of this solution it is necessary and sufficient that the inequality

$$s_1(0) \leqslant s_2(0) \tag{17}$$

be satisfied.

Analogous conditions can be given also for the periodic solution $Y_2(x)$ of equation (13). So, the fulfillment of inequality

$$r_1(\eta_0) > r_2(\eta_0) \tag{18}$$

is a necessary and sufficient condition for its nonexistence, while for the existence of $Y_2(x)$ it is necessary and sufficient that the inequality

$$r_1(\eta_0) \leqslant r_2(\eta_0) \tag{19}$$

be satisfied.

Criteria for the existence or nonexistence of periodic solutions with respect to x can be obtained by means of inequalities (6) to (19), using estimates for separatrices contained in these inequalities.

In order to derive conditions for the existence of a periodic solution $X(t)$ with respect to t of equation (3), it is necessary to consider the mutual disposition of the separatrices r_0, r_2, s_1, s_0. The disposition of the separatrices will be given by the comparison of the segments cut off by them either on the y-axis or on the x-axis. Denote by $x(s_0)$ the length of the segment from the origin of the coordinates along the x-axis to the largest positive root of the function $s_0(x)$, by $x(r_0)$ the length of the segment along the x-axis from the origin of coordinates to the smallest negative root of the function $r_0(x)$, and by $x(r_1)$ and $x(s_1)$ the lengths of the analogous segments. The points $x(r_0)$, $x(r_1)$, $x(s_1)$ and $x(s_0)$ are assumed to lie on the segment $[\eta_2, x_1]$.

Lemma 1. Let the bounded region D contain in its interior an unstable segment $[\eta_0, 0]$ of system (4) and let it be situated in the strip $x_2 \leqslant x \leqslant \eta_1$.

If there is a point on the boundary of the region D such that the positive half-trajectory issuing from this point lies in the region D and does not coincide with the separatrices s_1 and r_2, then there exists at least one stable limit cycle of system (4) embracing the segment $[\eta_0, 0]$.

Here and in what follows a limit cycle is called stable if it is stable in the sense of Liapunov.

Let us prove the Lemma. If a point, moving along a trajectory of system (4), remains as $t \to +\infty$ in the bounded region, then it must have a set of ω limit points, not intersecting the unstable segment $[\eta_0, 0]$ of this system. Since the set of ω limit points consists of complete trajectories of the system, then it must contain the limit cycle [2], necessarily embracing the unstable segment $[\eta_0, 0]$.

The limit cycle cannot lie in the upper half-plane. In fact, consider the function

$$v(x, y) = y^2 + 2\int_0^x f_1(x)\,dx.$$

Computing the derivative with respect to time, we have, due to equations (4)

$$dv/dt = -yR(x, y) < 0 \qquad \text{for } y > 0.$$

From this we conclude that the function v decreases along the limit cycle as t increases. This, however, contradicts the single-valuedness of function v.

If we assume that the limit cycle lies in the lower half-plane, then we arrive at an analogous contradiction by considering the function

$$v(x, y) = y^2 + 2\int_{\eta_0}^{x} f_2(x)\,dx \qquad \text{for } y \leqslant 0$$

On the other hand, the limit cycle cannot lie either to the left of the point $x = \eta_0$ nor to the right of the point $x = 0$, since such a disposition would not be consistent with the direction field of system (4). Hence, the limit cycle lies in the region D and embraces the segment $[\eta_0, 0]$.

If the found limit cycle is unstable, then inside it must lie another limit cycle, being the set of ω-limit points for points which recede from the first cycle as t increases. Using the method of transfinite induction, we obtain at least one stable limit cycle, lying in the region D and embracing the unstable segment $[\eta_0, 0]$ of the system.

Theorem 1. In order that system (4) possess at least one stable limit cycle it is sufficient that one of the following conditions be satisfied:

(a) $$0 < x(s_0) < x(r_2) \tag{20.1}$$

(b) $$x(s_1) < x(r_0) < \eta_0 \tag{20.2}$$

(c) $$s_0(0) < s_1(0), \qquad r_0(\eta_0) > r_2(\eta_0) \tag{20.3}$$

Let the condition (20.1) be satisfied (Fig. 4).

Consider the region D, bounded by the curves s_0, r_2 and the segment $[x(s_0), x(r_2)]$ of the x-axis. The segment $[\eta_0, 0]$ is situated inside the region D. The trajectory of system (4), coinciding with the separatrix s_0, enters the region D for some value of t. As t increases this trajectory cannot leave this region due to the structure of the direction field and uniqueness of the integral curves of system (4).

From the fulfillment of the conditions of Lemma 1 follows the existence of at least one stable limit cycle of the system.

In the case of the fulfillment of conditions (20.2) it is necessary to take for the region D the region bounded by the separatrices r_0, s_1

and the segment $[x(s_1), x(r_0)]$ of the x-axis, and for the trajectory to be investigated to take the trajectory of the system coinciding with the separatrix r_0.

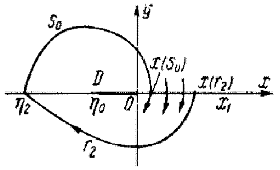

Fig. 4.

Finally, if the conditions (20.3) of the theorem are satisfied, one can take for the region D the region bounded by the separatrices s_0, s_1, r_2, r_0 and the segments $[r_2(\eta_0), r_0(\eta_0)]$, $[s_0(0), s_1(0)]$ of the y-axis.

Investigating the behavior of the trajectories of system (4), coinciding either with the separatrix s_0, or with the separatrix r_0, we can easily establish by Lemma 1 the existence of at least one stable limit cycle of system (4).

Let us consider in detail the derivation of sufficient criteria for the existence of a limit cycle of system (4).

Lemma 2. The integral curves s_0 and s of the system satisfy the inequalities

$$s_1(x) > \left(2\int_x^{x_1} f_1(x)\,dx\right)^{1/2}, \qquad s_0(x) < \left(2\int_x^{\eta_2} f_1(x)\,dx\right)^{1/2} \tag{21}$$

for all those x for which the corresponding integrals are positive.

In order to prove the first inequality consider the curve

$$y = \left(2\int_x^{x_1} f_1(x)\,dx\right)^{1/2} \tag{22}$$

This curve intersects the x-axis in addition to the point $x = x_1$, at a certain point $x = x_*$, situated on the segment $[x_2, 0]$. In fact, introducing the notation

$$\Phi(x) = 2\int_x^{x_1} f_1(x)\,dx$$

we have by virtue of the properties of the function $f_1(x)$ the inequalities $\Phi(0) > 0$ and $\Phi(x_2) \leqslant 0$, i.e. on the segment $[x_2, 0]$ there is a zero of the function $\Phi(x)$.

The separatrix s_1 satisfies the differential equation (13), i.e. the relation

$$s_1^2(x) = 2\int_x^{x_1} [R(x, s_1(x)) + f_1(x)]\,dx \tag{23}$$

Consider the difference of the squares between the ordinates of the curves s_1 and (22):

$$s_1^2(x) - y^2(x) = 2\int_x^{x_1} R(x, s_1(x))\,dx$$

For $s_1(x) > 0$ this difference is positive by virtue of the properties (10) of the function $R(s, y)$. In this way the validity of inequality (21) is proved for all $x_* < x < x_1$.

Analogously, the validity of inequality (22) for all those x can be proved, for which the curve

$$y = \left(2\int_x^{\eta_2} f_1(x)\,dx\right)^{1/2} \tag{24}$$

is situated above the x-axis. In exactly the same way also the following lemma can be proved.

Lemma 3. The integral curves r_0 and r_1 of system (12) satisfy the inequalities

$$r_2(x) < -\left(2\int_x^{\eta_2} f_2(x)\,dx\right)^{1/2}, \qquad r_0(x) > -\left(2\int_x^{x_1} f_2(x)\,dx\right)^{1/2} \tag{25}$$

for all those x for which the curves

$$y = -\left(2\int_x^{\eta_2} f_2(x)\,dx\right)^{1/2} \tag{26}$$

$$y = -\left(2\int_x^{x_1} f_2(x)\,dx\right)^{1/2} \tag{27}$$

respectively are situated below the x-axis.

Remark. It is easy to verify that for the values of x under consideration the curves (22), (24), (26), (27) are without contact [2] and intersect, as t increases, with the trajectories of system (4) along the x-axis. In fact, consider the function

$$v(x, y) = y^2 - 2\int_x^{x_1} f_1(x)\,dx$$

For this function the derivative with respect to time, calculated with due regard to equations (4),

$$dv/dt = -2yR(x, y)$$

is negative for all $y > 0$ according to the properties (10) of the function $R(x, y)$.

Noticing that $v(0, 0) < 0$, we conclude that the curve (22) is contactless.

Analogously one can verify the fact that the other curves are also contactless.

Theorem 2. If the curves (24) and (26), issuing from the point $(\eta_2, 0)$, intersect the x-axis correspondingly at the points $x = x_*$, $x = \eta_*$ and $0 < x_* < \eta_* < \eta_1$, then the system (4) has at least one stable limit cycle.

In order to prove this theorem, consider the region D, bounded by the curves (24, (26) and the segment $[x_*, \eta_*]$ of the x-axis. Since the curves (24) and (26) are without contact, and are intersected by the trajectories of system (4) in the inward direction of the region, as is the segment $[x_*, \eta_*[$, the trajectories of system (4), intersecting the boundary of the region D, as t increases, enter the region and remain in it as $t \to +\infty$. The region D so constructed, embraces the segment $[\eta_0, 0]$ of unstable points of the system. Applying Lemma 1 we convince ourselves of the existence of at least one stable limit cycle of system (4).

Theorem 3. If the curves (22) and (27), issuing from the point $(x_1, 0)$, intersect the x-axis at the points $x = x_*$ and $x = \eta_*$ in such a way that the inequalities $x_2 < x_* \leqslant \eta_* < \eta_0$ are satisfied, then the system (4) has at least one stable limit cycle.

The proof of this theorem can be given analogously to the proof of the preceding theorem. It is necessary to take for the region D the region bounded by the contactless curves (22) and (27) and the segment $[x_*, \eta_*]$ of the x-axis.

Let us remark that Theorems 2 and 3 can also be proved by means of checking correspondingly the conditions (20.1) and (20.2) of Theorem 1.

Theorem 4. If

$$2\int_0^{x_1} f_1(x)\,dx \geqslant A^2, \qquad 2\int_{\eta_0}^{\eta_2} f_2(x)\,dx \geqslant B^2$$

where

$$A = \max \Phi_1(x) \quad \text{for } \eta_2 \leqslant x \leqslant 0, \qquad B = \min \Phi_2(x) \quad \text{for } \eta_0 \leqslant x \leqslant x_1$$

are the extremal values for the ordinates of the isoclinics of zero inclination (14) in the corresponding intervals of variation of x, then the system (4) possesses at most one stable limit cycle.

In fact, from the properties of isoclinics (14), it follows that $s_0(0) < A$ and $r_0(\eta_0) > B$. On the other hand, the curve (22) is contactless and lies below the separatrix s_1, in any case for $0 \leqslant x < x_1$; the curve (26) is above the separatrix r_2 for $\eta_2 < x \leqslant \eta_0$. Consequently, we have

$$s_1(0) > \Big(2\int_0^{x_1} f_1(x)\,dx\Big)^{1/2}, \qquad r_2(\eta_0) < -\Big(2\int_{\eta_0}^{\eta_2} f_2(x)\,dx\Big)^{1/2}$$

in conformity with inequalities (21) of Lemma 2. Comparing the obtained inequalities with the conditions of the theorem to be proved, we conclude that from the fulfillment of the conditions of the theorem follows the validity of inequalities (c) of Theorem 1. Thus, system (4) has in fact at most one limit cycle.

Applying Theorem 4 to equation (1) we can assert that for sufficiently large (a) this equation has a periodic solution with respect to t, corresponding to the undamped oscillations of a pendulum.

Theorem 5. If there exist values $x = x_*$ and $x = \eta_*$ on the segments $[0, x_1]$ and $[\eta_2, \eta_0]$, respectively, such that the inequalities

$$\int_{\eta_*}^{x_*} f_2(x)\,dx \leqslant 0, \qquad \int_{\eta_*}^{x_*} f_1(x)\,dx \geqslant 0$$

are satisfied simultaneously, then system (4) possesses at least one stable limit cycle.

In order to prove the theorem consider the curves

$$y = \Big(2\int_{x}^{\eta_*} f_1(x)\,dx\Big)^{1/2} \tag{28}$$

$$y = -\Big(2\int_{x}^{x_*} f_2(x)\,dx\Big)^{1/2} \tag{29}$$

Let

$$F(x) = \int_{x}^{\eta_*} f_1(x)\,dx \tag{30}$$

Then by virtue of the assumptions with respect to the function $f_1(x)$ we have $F(0) > 0$ and $F(x_*) \leqslant 0$, i.e. the curve (28) intersects the x-axis at a certain point $x = \xi$ on the segment $[0, x_*]$ as well as at the point $x = \eta_*$. Analogously, it can be established that the curve (29) intersects the x-axis at a certain point $x = \zeta$ of the segment $[\eta_*, \eta_0]$ as well as at the point $x = x_*$. Consider the region bounded by the curves (28) and (29) and the segments $[\eta_*, \zeta]$ and $[\xi, x_*]$.

It is easy to show that the curves (28) and (29) are contactless curves, intersected by the trajectories of system (4) inside the region D as t increases. By virtue of the structure of the direction field of system (4) the segments $[\eta_*, \zeta]$ and $[\xi, x_*]$ are also intersected by the trajectories of the system, entering the region D as t increases. Finally, from the conditions of our theorem follows that the unstable segment $[\eta_0, 0]$ is contained inside the region D.

From the above considerations follows that for the region D the conditions of Lemma 1 are satisfied, i.e. the system (4) actually possesses one stable limit cycle, if the assumptions of our theorem are satisfied.

Corollary. If the inequalities

$$\int_{\eta_2}^{x_1} f_2(x)\,dx \leqslant 0, \qquad \int_{\eta_2}^{x_1} f_1(x)\,dx \geqslant 0 \tag{31}$$

are satisfied simultaneously, then system (4) possesses at least one stable limit cycle.

The correctness of the formulated statement follows from Theorem 5 proved above for $x_* = x_1$ and $\eta_* = \eta_2$.

Notice, however, that the corollary to Theorem 5 can also be proved independently by verifying that the inequalities (20.3) of Theorem 1 are satisfied.

In conclusion let us remark that the estimates for the separatrices, given by Lemma 2, can be improved without great difficulties. The sufficient conditions for the existence of limit cycles of system (4), formulated in Theorems 2 to 5, in the same way, can be made more precise; it is true that this can be achieved only by making them more complicated.

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